Classical & Quantum Waves

Dispersion of waves

- So far we have considered cases where the velocity of a wave is independent of frequency:
  - Acoustic waves in air
  - EM waves in vacuum
  - Waves on taut string $v = \sqrt{\frac{T}{\mu}}$ (indep. of ω!)
- In general, waves in a medium show "dispersion", i.e., the velocity depends on frequency $v(\omega)$
- The medium in this case is called a "dispersive medium"

→ relevant for waves composed of ~~superposition~~ group of many waves w/ different frequencies

- Familiar example is the separation of ~~of~~ white light into colors of rainbow in a prism

↳ happens b/c velocity of light is frequency-dependent in glass

- In cases w/ groups of waves of different ~~velocities~~ frequencies, their superposition leads to a "modulated" wave

- In dispersive media, individual waves in group travel at different velocities "phase velocity", which causes the modulation to move at a velocity called the "group velocity"

  ↳ different from individual waves in the group

- We formalize this below:

Superposition of waves in dispersive media

- A simple monochromatic wave $\psi = A\cos(kx-\omega t)$ cannot carry useful information (eg, for radio, cell phone etc). Modulated waves are needed to send information

  → superposition of waves

Beats

- Consider superposition of 2 monochromatic waves

$$\psi_1 = A\cos(k_1x-\omega_1 t) \qquad \psi_2 = A\cos(k_2x-\omega_2 t)$$

- In a non-dispersive medium, both waves travel at same velocity $v = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$

- Superposition of 2 waves gives: $\psi_1+\psi_2 = A\cos(k_1x-\omega_1 t)+A\cos(k_2x-\omega_2 t)$

- We will put this in a different form by using the trig identity: $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

- let $(\alpha+\beta) = k_2 x - \omega_2 t$ $\quad\quad (\alpha-\beta) = k_1 x - \omega_1 t$

- need to find α, β

$\underline{\alpha}$

→ take $(\alpha+\beta) + (\alpha-\beta) = 2\alpha = k_2 x - \omega_2 t + k_1 x - \omega_1 t$

$$\Rightarrow \alpha = \frac{1}{2}\left[(k_2+k_1)x - (\omega_2+\omega_1)t\right]$$

$\underline{\beta}$

→ take $(\alpha+\beta) - (\alpha-\beta) = 2\beta = k_2 x - \omega_2 t - (k_1 x - \omega_1 t)$

$$\Rightarrow \beta = \frac{1}{2}\left[(k_2-k_1)x - (\omega_2-\omega_1)t\right]$$

$$\Rightarrow \psi = 2A\cos\alpha\cos\beta = 2A\cos\left[\frac{(k_2+k_1)}{2}x - \frac{(\omega_2+\omega_1)}{2}t\right]\cos\left[\frac{(k_2-k_1)}{2}x - \frac{(\omega_2-\omega_1)}{2}t\right]$$

- Let's first consider what happens at a fixed position $x=0$

$$\Rightarrow \boxed{\psi = 2A\cos\left[\frac{(\omega_2-\omega_1)}{2}t\right]\cos\left[\frac{(\omega_2+\omega_1)}{2}t\right]}$$

- ~~Interesting~~ This is a general statement, but interesting case when considering $\omega_1 \approx \omega_2$, b/c then 1st term becomes a slow amplitude modulation on 2nd term

$\psi(t) = A(t)\cos\omega_0 t$ $\quad$ w/ $A(t) = 2A\cos\left[\frac{(\omega_2-\omega_1)}{2}t\right]$, $\omega_0 = \frac{\omega_2+\omega_1}{2}$

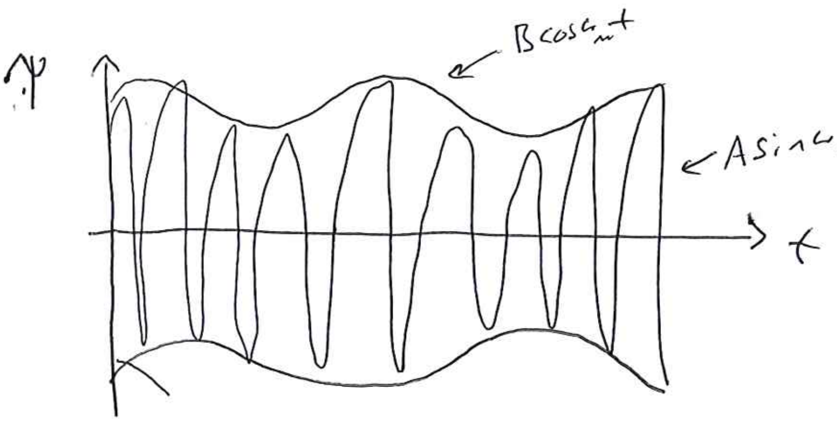
Amplitude modulation (AM)

- AM used to encode information in traveling EM waves, e.g. AM radio
- Idea: modulate the amplitude of EM wave, called "carrier wave", to carry information such as speech or music

$$\psi = (\underbrace{A}_{\text{depth of modulation}} + B\cos \underbrace{\omega_m}_{\text{modulation frequency}} t)\sin\omega_c t$$

ω_c: carrier wave frequency

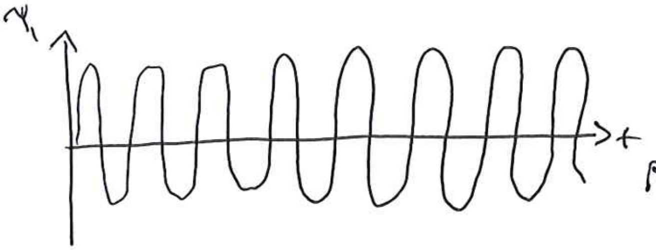
$B \ll A$ for practical purposes, to avoid distortion at receiver



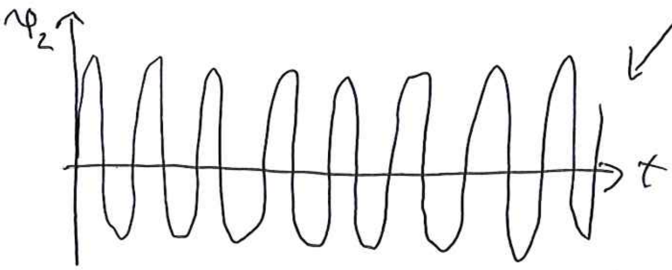
- Rewrite ψ using trig identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

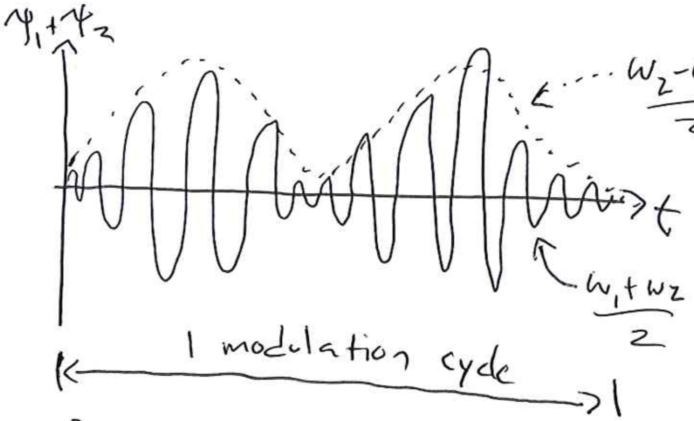
No proof, but we are essentially doing the opposite process of what we did earlier in the lecture

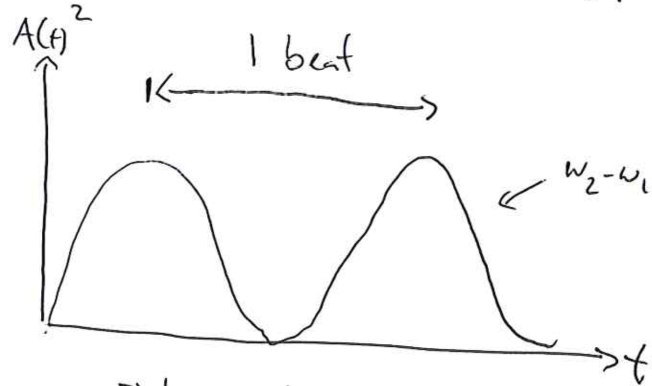
$$\psi = A\sin\omega_c t + \frac{B}{2}[\sin(\omega_c + \omega_m)t + \sin(\omega_c - \omega_m)t]$$



slightly different frequencies ω_1, ω_2







→ beats happen at twice the modulation frequency ($\frac{1}{2}$ the period)

Example

· Take 2 tuning forks at slightly different frequencies.

We hear a well-defined central frequency $\frac{(\omega_1 + \omega_2)}{2}$ and a slow modulation of the volume on top ~~at $(\omega_2 - \omega_1)$~~ corresponding to $A(t)^2$ at $(\omega_2 - \omega_1)$

→ $\omega_1 = 439$ Hz $\quad \omega_2 = 441$ Hz
[book typo here]

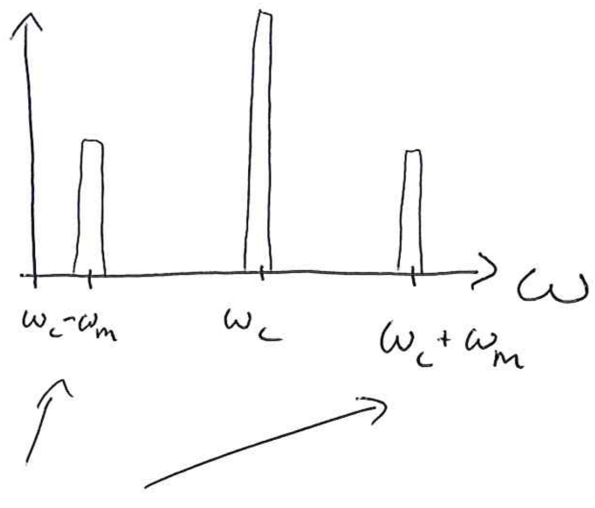
→ hear $\frac{\omega_1 + \omega_2}{2} = 400$ Hz

w/ modulation at 2 Hz

→ Beats are used to tune stringed instruments, tune string until beat goes away when compared to another string that is already in tune

- Inspection of this eq. reveals three frequencies

ω_c , $\omega_c + \omega_m$, $\omega_c - \omega_m$



- In radio, ω_m takes on (audio) range from ~ 10 Hz $\to 10$ kHz

- carrier wave $\omega_c \sim 1$ MHz

"side bands". In radio, there are may side bands (one for each ω_m frequency)

- Radio receivers "demodulate" AM wave to ~~the~~ give ω_m's by itself, which contains audio content

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<u>Dispersion of waves</u>

- In non-dispersive medium, velocity of wave is independent of wave number k

$$\omega = \text{constant} \times k \qquad v = \frac{\omega}{k}$$

↑ velocity (Freq. independent)

- In dispersive medium, velocity <u>does</u> depend on k, and so will the frequency $\omega = v k$

- relation between frequency & wavenumber called "dispersion relation"

· Dispersion relation is determined by physical properties of medium. Different media in general have different dispersion relations & wave behavior.

· In dispersive media, ω depends on k $\rightarrow$ $\omega = \omega(k)$

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Phase & group velocities

· Again consider 2 monochromatic waves

$$\psi_1 = A\cos(k_1 x - \omega_1 t) \quad , \quad \psi_2 = A\cos(k_2 x - \omega_2 t)$$

w/ identical amplitudes but slightly different frequencies $\omega_1 \approx \omega_2$

· Superposition is: $\psi = 2A\cos\left[\frac{(k_2-k_1)}{2}x - \frac{(\omega_2-\omega_1)}{2}t\right]\cos\left[\frac{(k_2+k_1)}{2}x - \frac{(\omega_2+\omega_1)}{2}t\right]$

[refer to earlier this lecture]

· The difference considered here is that now the medium is dispersive, so the 2 waves have different velocities:

$$V_1 = \frac{\omega_1}{k_1} \quad , \quad V_2 = \frac{\omega_2}{k_2}$$

· Let $k_0 \equiv \frac{k_2+k_1}{2}$, $\omega_0 \equiv \frac{\omega_2+\omega_1}{2}$ mean values of ω, k

$\Delta k = \frac{k_2-k_1}{2}$, $\Delta\omega \equiv \frac{\omega_2-\omega_1}{2}$ differences of ω, k

$$\Rightarrow \quad \psi = A(x,t)\cos(k_0 x - \omega_0 t)$$

$$w/ \quad A(x,t) = 2A\cos(x\Delta k - t\Delta\omega)$$

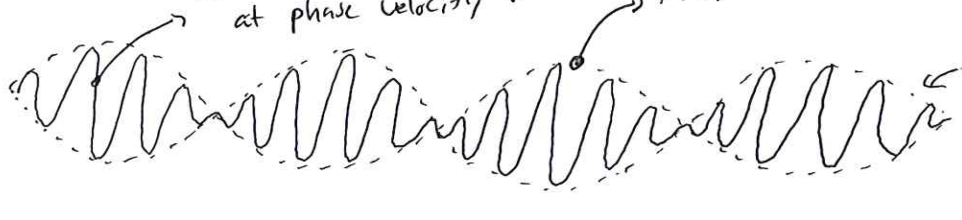
- This describes a wave of frequency ω_0, wavenumber k_0 and velocity $v = \dfrac{\omega_0}{k_0}$

  → this is called the "phase velocity". It's the speed the carrier wave is moving at.

- But there is another important velocity to consider, which is the velocity that the modulated disturbance is traveling at → "group velocity"

  [See Fig. 8.3 King]

- This is the speed at which the amplitude modulation $A(x,t)$ travels at



- The amplitude of the crest of the envelope remains constant as the ~~wave~~ envelope propagates on top of the carrier wave

$$[A(x,t) = 2A\cos(x\Delta k - t\Delta\omega)]$$

$$\rightarrow A(x,t) = \text{constant}$$

$$\rightarrow x\Delta k - t\Delta\omega = \text{constant}$$

- Differentiating this eq. wrt time gives the velocity the envelope travels at, v_g.

$$\frac{d}{dt}\left[x\Delta k - t\Delta\omega\right] = \underbrace{\frac{d}{dt}[\text{constant}]}_{=0}$$

$$\rightarrow \frac{dx}{dt}\Delta k = \Delta\omega \quad \Rightarrow v_g = \frac{dx}{dt} \simeq \frac{\Delta\omega}{\Delta k} = \frac{\omega_2 - \omega_1}{k_2 - k_1}$$

"group velocity"

- Since ω is a function of k, we write:

$$\cancel{v_g = \frac{\omega_2 - \omega_1}{k}} \qquad v_g = \frac{\omega(k_2) - \omega(k_1)}{k_2 - k_1} \qquad \text{eq. 8.18}$$

- We want to find expression for $\omega(k_2)-\omega(k_1)$
- Using Taylor's theorem, we can write:

$$\omega(k_0 \pm \Delta k) = \omega(k_0) \pm (\Delta k)\left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}$$

$$\text{w/ } \Delta k = \frac{k_2 - k_1}{2}, \quad k_0 = \frac{k_2 + k_1}{2}$$

+ case

$$\Rightarrow \omega\Big(\underbrace{\frac{k_2+k_1}{2} + \frac{k_2-k_1}{2}}_{=k_2}\Big) = \omega(k_0) + \left(\frac{k_2-k_1}{2}\right)\left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}$$

− case

$$\Rightarrow \omega\Big(\underbrace{\frac{k_2+k_1}{2} - \frac{(k_2-k_1)}{2}}_{=k_1}\Big) = \omega(k_0) - \left(\frac{k_2-k_1}{2}\right)\left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}$$

$$\Rightarrow \omega(k_2) - \omega(k_1) = (k_2 - k_1)\left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}$$

- recall: $V_g = \dfrac{\omega(k_2) - \omega(k_1)}{k_2 - k_1}$

plug in $\omega(k_2)-\omega(k_1)$ $\Rightarrow$ $\boxed{V_g = \left(\frac{\partial \omega}{\partial k}\right)_{k=k_0}}$ universal expression for group velocity [velocity of envelope]

vs. phase velocity $\boxed{v = \frac{\omega}{k}}$ [velocity of carrier wave]

or V_{ph}